# Conformal Yano-Killing tensors for the Taub-NUT metric

Jacek Jezierski* and Maciej Łukasik
Department of Mathematical Methods in Physics,
University of Warsaw, ul. Hoża 69, 00-681 Warsaw, Poland

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**Abstract**

Symmetric conformal Killing tensors and (skew-symmetric) conformal Yano–Killing tensors for Euclidean Taub-NUT metric are given in explicit form. Relations between Yano and CYK tensors in terms of conformal rescaling are discussed.

## 1 Introduction

In [13] we examined conformal Yano–Killing tensors in Kerr spacetime. In this paper we discuss Euclidean Taub-NUT metric which is also an interesting case possessing non-trivial CYK tensors.

According to [12] one can define, in terms of spacetime curvature, two kinds of conserved quantities with the help of conformal Yano–Killing tensors (see [20], [21]). Sometimes they are also called conformal Killing forms or twistor forms (see e.g. [14], [17], [18]). The first kind is linear and the second quadratic with respect to the Weyl tensor but a basis for both of them is the Maxwell field. Conserved quantities which are linear with respect to CYK tensor were investigated many times (cf. [6], [7], [9], [10], [12], [15], [16]). On the other hand, quadratic charges are less known and have usually been examined in terms of the Bel–Robinson tensor (see e.g. [1], [2], [3], [4]).

This paper is organized as follows: In the next Section we introduce basic notions, CYK tensors for Euclidean Taub-NUT metric and derive conformal symmetric Killing tensors. In Section 3 we analyze the question *if we can reduce Conformal Yano-Killing tensor to Yano by conformal transformation?*

## 2 Taub-NUT metrics and its CYK tensors

Let $M$ be an $n$-dimensional ($n > 1$) manifold with a Riemannian or pseudo-Riemannian metric $g_{\mu\nu}$. The covariant derivative associated with the Levi–Civita connection will

*Partially supported by EPSRC: EP/D032091/1. E–mail: Jacek.Jezierski@fuw.edu.pl

be denoted by $\nabla$ or just by "$;$". By $T_{\dots(\mu\nu)\dots}$ we will denote the symmetric part and by $T_{\dots[\mu\nu]\dots}$ the skew-symmetric part of tensor $T_{\dots\mu\nu\dots}$ with respect to indices $\mu$ and $\nu$ (analogous symbols will be used for more indices).

Let $Q_{\mu\nu}$ be a skew-symmetric tensor field (two-form) on $M$ and let us denote by $\mathcal{Q}_{\lambda\kappa\sigma}$ a (three-index) tensor which is defined as follows:

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) . \tag{2.1}$$

The object $\mathcal{Q}$ has the following algebraic properties

$$\mathcal{Q}_{\lambda\kappa\mu}g^{\lambda\mu} = 0 = \mathcal{Q}_{\lambda\kappa\mu}g^{\lambda\kappa} , \quad \mathcal{Q}_{\lambda\kappa\mu} = \mathcal{Q}_{\mu\kappa\lambda} , \tag{2.2}$$

i.e. it is traceless and partially symmetric. In [13] (see also [9], [10]) we proposed the following

**Definition 1** *A skew-symmetric tensor $Q_{\mu\nu}$ is a conformal Yano–Killing tensor (or simply CYK tensor) for the metric $g$ iff $\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) = 0$.*

In other words, $Q_{\mu\nu}$ is a conformal Yano–Killing tensor if it fulfils the following equation:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) \tag{2.3}$$

(first proposed by Tachibana and Kashiwada, cf. [20]). Moreover, if $\xi_\mu := Q^{\nu}{}_{\mu;\nu}$ vanishes then $Q$ is a usual Yano tensor i.e. a solution of equation (2.3) with vanishing right-hand side.

Let us consider Euclidean Taub-NUT metric which is an example of a metric admitting nontrivial solutions of the equation (2.3). We will define it in terms of coordinate system $(\psi, r, \theta, \phi)$. In these coordinates the metric tensor has a form:

$$g = \left(1 + \frac{2m}{r}\right)\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right) + \frac{4m^2}{1+\frac{2m}{r}}\left(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right)^2 . \tag{2.4}$$

Passing to the limit as $r \to \infty$ we get:

$$g = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 + \mathrm{d}\bar{\psi}^2, \tag{2.5}$$

where $\bar{\psi} := 2m\psi$ (the term $\cos\theta\,\mathrm{d}\phi$ is negligible relative to $r\sin\theta\,\mathrm{d}\phi$). It means that for large $r$ the metric $g$ looks like flat Euclidean metric and therefore may be called "asymptotically flat Euclidean metric". However, Taub-NUT manifold (denoted by $M$) is a bundle $M \to S^2$ with base coordinates $(\theta, \phi)$ on the two-sphere $S^2$. Fibre coordinates are $(\psi, r)$. Moreover, it is not a trivial bundle, i.e. it has no global section. If, for example, we restrict ourselves to points with constant $\psi$ and $r$, than the metric induced on this surface is singular. We will see that it causes some difficulties.

Similarly to the case of Kerr metric (cf. [13]) there are known Yano tensors for the metric (2.4) (see e.g. [8]). They are given by the following formulae:

$$Y = 2m^2\left(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right) \wedge \mathrm{d}r + r\,(r+m)\,(r+2m)\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{2.6}$$

$$
Y_i = 4m\,(\,\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi) \wedge\, \mathrm{d}x_i - (1 + \frac{2m}{r})\epsilon_{ijk}\,\mathrm{d}x^j \wedge\, \mathrm{d}x^k, \tag{2.7}
$$

where $1 \le i,\, j,\, k \le 3$. The functions $x^i = x_i$ and the symbol $\epsilon_{ijk}$ are defined by:

$$
x^1 := r\sin\theta\cos\phi,
$$

$$
x^2 := r\sin\theta\sin\phi,
$$

$$
x^3 := r\cos\theta,
$$

$$
\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of 1,2,3} \\ -1 & \text{if } ijk \text{ is an odd permutation of 1,2,3} \\ \phantom{+}0 & \text{in any other cases} \end{cases}
$$

We can now ask, how tensors dual to $Y$ and $Y_i$ look like. It turns out that tensors $Y_i$ are anti-selfdual, i.e. $*Y_i = -Y_i$. They are also covariantly constant, i.e. $\nabla_\rho (Y_i)_{\mu\nu} = 0$ Moreover, the two-forms $F_{\mu\nu}(R, Y_i) := R_{\mu\nu\lambda\kappa}(Y_i)^{\lambda\kappa}$, where $R$ is a Riemann tensor[1] of the metric $g$, are identically equal to zero.

Tensor dual to $Y$ has a form:

$$
*\,Y = 2m\,(m+r)\,(\,\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi) \wedge\, \mathrm{d}r + mr\,(r+2m)\sin\theta\,\mathrm{d}\theta \wedge\, \mathrm{d}\phi. \tag{2.8}
$$

$*Y$ is not a Yano tensor anymore. Its divergence $\chi$ defined as $\chi^\nu := *Y^{\mu\nu}{}_{;\mu}$ equals to $-\frac{3}{2m}\partial_\psi$, which implies that equation (2.3) for $*Y$ has nontrivial right-hand side.

We may try to derive "Euclidean" charges corresponding to tensors $Y$ and $*Y$ . Asymptotically they look as follows:

$$
Y = r^3\sin\theta\,\mathrm{d}\theta \wedge\, \mathrm{d}\phi + O\,(1) = *(r\,\mathrm{d}\psi \wedge\, \mathrm{d}r) + O\,(1)\,,
$$

$$
*Y = r\,\mathrm{d}\psi \wedge\, \mathrm{d}r + O\,(1)\,.
$$

We can say, then, that a charge corresponding to $*Y$ is the (Euclidean) energy, and a charge corresponding to $Y$ is the dual energy (cf. [13]).

It turns out that, although $Y$ and $*Y$ are different tensors, the forms $F(R, Y)$ and $F(R, *Y)$ are the same. We will denote it by $\widetilde{F} := F(R, Y) = F(R, *Y)$. We have:

$$
\widetilde{F} = \frac{8m^2}{(r+2m)^2}\,\mathrm{d}\psi \wedge\, \mathrm{d}r + \frac{4rm\sin\theta}{r+2m}\,\mathrm{d}\theta \wedge\, \mathrm{d}\phi + \frac{8m^2\cos\theta}{(r+2m)^2}\,\mathrm{d}\phi \wedge\, \mathrm{d}r. \tag{2.9}
$$

The fact $F(R, Y) = F(R, *Y)$ implies that charges corresponding to $Y$ and $*Y$ are the same, energy is self-dual i.e. energy and dual energy are equal. Unfortunately in the case of Taub-NUT metric, we cannot define the charge as a integral over a closed surface, since our spacetime $M$ is nontrivial bundle $M \to S^2$ and there is no

[1] Ricci tensor of Taub-NUT metric vanishes, which implies that its Riemann tensor is a spin-2 field.

"sphere at infinity". However, we may project the form $\widetilde{F}^{\mu\nu}\,\mathrm{d}S_{\mu\nu}$ onto the base of the fibration, integrate it, and get the result, which depends on $r$:

$$\frac{1}{16\pi}\int_{S^2}\widetilde{F}^{\mu\nu}\,\mathrm{d}S_{\mu\nu}=\frac{mr}{r+2m}\xrightarrow[r\to\infty]{}m.$$

It is not surprising that the "dual mass" cannot be defined as a integral over a finite two-surface.

For $\widetilde{F}$ there exist a potential $\widetilde{A}$ ($\widetilde{F}=\mathrm{d}\widetilde{A}$) equal to

$$\widetilde{A}=-\frac{4mr}{r+2m}\,\mathrm{d}\psi-\frac{4mr\cos\theta}{r+2m}\,\mathrm{d}\phi, \tag{2.10}$$

$$g^{-1}(\widetilde{A})=-\frac{1}{m}\partial_\psi. \tag{2.11}$$

We see that for such a potential $g^{-1}(\widetilde{A})$ is a Killing vector field of the metric $g$, and therefore $\widetilde{F}$ is a Papapetrou field (like $F(R,*Q)$ for the Kerr metric cf. [13]).

It remains to check, how the conformal Killing tensor related to $Y$, $*Y$, and $Y_i$ look like. Let us introduce the following notation:

$$K_{\mu\nu}(Y,Z):=\frac{1}{2}\left(Y_\mu{}^\kappa Z_{\kappa\nu}+Z_\mu{}^\kappa Y_{\kappa\nu}\right).$$

In coordinate system $(\psi,r,\theta,\phi)$ the easiest way of writing out these tensors is to treat them as covariant tensors (with lowered indices). We have the following non-vanishing components of the symmetric $5\times5$ matrix $K(\cdot,\cdot)$:

$$K(Y,Y)=-\frac{1}{1+\frac{2m}{r}}\Big[4m^4\,\mathrm{d}\psi^2+8m^4\cos\theta\,\mathrm{d}\psi\,\mathrm{d}\phi+(m+r)^2(r+2m)^2\,\mathrm{d}\theta^2+$$
$$+\left(r\sin^2\theta(3m+r)(4m^2+3mr+r^2)+4m^2\right)\,\mathrm{d}\phi^2+m^2(1+\frac{2m}{r})^2\,\mathrm{d}r^2\Big],$$

$$K(Y_i,Y_j)=-\frac{1}{1+\frac{2m}{r}}\Big[16m^2\,\mathrm{d}\psi^2+32m^2\cos\theta\,\mathrm{d}\psi\,\mathrm{d}\phi+4(r+2m)^2\,\mathrm{d}\theta^2+$$
$$+4\left(r(r+4m)\sin^2\theta+4m^2\right)\,\mathrm{d}\phi^2+\frac{4(r+2m)^2}{r^2}\,\mathrm{d}r^2\Big]\delta_{ij}$$

$$K(Y,Y_1)=-\frac{1}{1+\frac{2m}{r}}\Big[8m^3\sin\theta\cos\phi\,\mathrm{d}\psi^2-2(m+r)(r+2m)^2\sin\theta\cos\phi\,\mathrm{d}\theta^2+$$
$$+2\Sigma\sin\theta\cos\phi\,\mathrm{d}\phi^2+\frac{2m(r+2m)^2\sin\theta\cos\phi}{r^2}\,\mathrm{d}r^2+$$
$$-2m(r+2m)^2\sin\phi\,\mathrm{d}\psi\,\mathrm{d}\theta-2mr(r+4m)\sin\theta\cos\theta\cos\phi\,\mathrm{d}\psi\,\mathrm{d}\phi+$$
$$-2m(r+2m)^2\cos\theta\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi+\frac{(r+2m)^3\cos\theta\cos\phi}{r}\,\mathrm{d}\theta\,\mathrm{d}r+$$

$$
\begin{aligned}
& -\frac{(r+2m)^3\sin\theta\sin\phi}{r}\,\mathrm{d}\phi\,\mathrm{d}r\Big] \\
K(Y,Y_2) &= -\frac{1}{1+\frac{2m}{r}}\Big[8m^3\sin\theta\sin\phi\,\mathrm{d}\psi^2 - 2(m+r)(r+2m)^2\sin\theta\sin\phi\,\mathrm{d}\theta^2 + \\
& +2\Sigma\sin\theta\sin\phi\,\mathrm{d}\phi^2 + \frac{2m(r+2m)^2\sin\theta\sin\phi}{r^2}\,\mathrm{d}r^2 \\
& +2m(r+2m)^2\cos\phi\,\mathrm{d}\psi\,\mathrm{d}\theta - 2mr(r+4m)\sin\theta\cos\theta\sin\phi\,\mathrm{d}\psi\,\mathrm{d}\phi \\
& +2m(r+2m)^2\cos\theta\cos\phi\,\mathrm{d}\theta\,\mathrm{d}\phi + \frac{(r+2m)^3\cos\theta\sin\phi}{r}\,\mathrm{d}\theta\,\mathrm{d}r + \\
& +\frac{(r+2m)^3\sin\theta\cos\phi}{r}\,\mathrm{d}\phi\,\mathrm{d}r\Big] \\
K(Y,Y_3) &= -\frac{1}{1+\frac{2m}{r}}\Big[8m^3\cos\theta\,\mathrm{d}\psi^2 - 2(m+r)(r+2m)^2\cos\theta\,\mathrm{d}\theta^2 + \\
& +2(\Sigma+2m(r+2m)^2)\cos\theta\,\mathrm{d}\phi^2 + \frac{2m(r+2m)^2\cos\theta}{r^2}\,\mathrm{d}r^2 + \\
& +2m(r^2\sin^2\theta + 4mr\sin^2\theta + 4m^2)\,\mathrm{d}\psi\,\mathrm{d}\phi - \frac{(r+2m)^3\sin\theta}{r}\,\mathrm{d}\theta\,\mathrm{d}r\Big] \\
K(Y,*Y) &= -\frac{1}{1+\frac{2m}{r}}\Big[4m^3(m+r)\,\mathrm{d}\psi^2 + m(m+r)(r+2m)^2\,\mathrm{d}\theta^2 + \\
& +m(m+r)(4m^2 + r(r+4m)\sin^2\theta)\,\mathrm{d}\phi^2 + \\
& +\frac{m(r+2m)^2(m+r)}{r^2}\,\mathrm{d}r^2 + 4m^3(m+r)\cos\theta\,\mathrm{d}\psi\,\mathrm{d}\phi\Big] \\
K(*Y,Y_1) &= -\frac{1}{1+\frac{2m}{r}}\Big[8m^2(m+r)\sin\theta\cos\phi\,\mathrm{d}\psi^2 - 2m(r+2m)^2\sin\theta\cos\phi\,\mathrm{d}\theta^2 + \\
& -2m(r^2\cos^2\theta + (r+2m)^2)\sin\theta\cos\phi\,\mathrm{d}\phi^2 + \frac{2(m+r)(r+2m)^2\sin\theta\cos\phi}{r^2}\,\mathrm{d}r^2 + \\
& -2m(r+2m)^2\sin\phi\,\mathrm{d}\psi\,\mathrm{d}\theta - 2mr^2\sin\theta\cos\theta\cos\phi\,\mathrm{d}\psi\,\mathrm{d}\phi + \\
& -2m(r+2m)^2\cos\theta\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi + \frac{(r+2m)^3\cos\theta\cos\phi}{r}\,\mathrm{d}\theta\,\mathrm{d}r + \\
& -\frac{(r+2m)^3\sin\theta\sin\phi}{r}\,\mathrm{d}\phi\,\mathrm{d}r\Big] \\
K(*Y,Y_2) &= -\frac{1}{1+\frac{2m}{r}}\Big[8m^2(m+r)\sin\theta\sin\phi\,\mathrm{d}\psi^2 - 2m(r+2m)^2\sin\theta\sin\phi\,\mathrm{d}\theta^2 + \\
& -2m(r^2\cos^2\theta + (r+2m)^2)\sin\theta\sin\phi\,\mathrm{d}\phi^2 + \frac{2(m+r)(r+2m)^2\sin\theta\sin\phi}{r^2}\,\mathrm{d}r^2 + \\
& +2m(r+2m)^2\cos\phi\,\mathrm{d}\psi\,\mathrm{d}\theta - 2mr^2\sin\theta\cos\theta\sin\phi\,\mathrm{d}\psi\,\mathrm{d}\phi + \\
& +2m(r+2m)^2\cos\theta\cos\phi\,\mathrm{d}\theta\,\mathrm{d}\phi + \frac{(r+2m)^3\cos\theta\sin\phi}{r}\,\mathrm{d}\theta\,\mathrm{d}r + \\
& +\frac{(r+2m)^3\sin\theta\cos\phi}{r}\,\mathrm{d}\phi\,\mathrm{d}r\Big] \\
K(*Y,Y_3) &= -\frac{1}{1+\frac{2m}{r}}\Big[8m^2(m+r)\cos\theta\,\mathrm{d}\psi^2 - 2m(r+2m)^2\cos\theta\,\mathrm{d}\theta^2 +
\end{aligned}
$$

$$+2m(4m^2+4mr+r^2\sin^2\theta)\cos\theta\,\mathrm{d}\phi^2+\frac{2(m+r)(r+2m)^2\cos\theta}{r^2}\,\mathrm{d}r^2+$$
$$+2m(4m^2+4mr+r^2\sin^2\theta)\,\mathrm{d}\psi\,\mathrm{d}\phi-\frac{(r+2m)^3\sin\theta}{r}\,\mathrm{d}\theta\,\mathrm{d}r\Big]$$

$$\begin{aligned}K(*Y,*Y) = -\frac{1}{1+\frac{2m}{r}}\Big[&4m^2(m+r)^2\,\mathrm{d}\psi^2+8m^2(m+r)^2\cos\theta\,\mathrm{d}\psi\,\mathrm{d}\phi+\\ &+m^2(4mr\cos^2\theta+3r^2\cos^2\theta+r^2+4mr+4m^2)\,\mathrm{d}\phi^2+\\ &+m^2(r+2m)^2\,\mathrm{d}\theta^2+\frac{(m+r)^2(r+2m)^2}{r^2}\,\mathrm{d}r^2\Big],\end{aligned}$$

where $\Sigma := -r^2(r+3m)\sin^2\theta - 2m(r^2+4mr+2m^2)$. Matrix $K(\cdot,\cdot)$ contains $4\times 4$ symmetric sub-matrix of Killing tensors (corresponding to Yano tensors $Y_i$ and $Y$) with 5 non-vanishing terms. The remaining 5 components are conformal Killing tensors for Taub-NUT metric.

# 3 Conformal rescaling of CYK tensors

In this section we will be dealing with conformal transformations and their impact on conformal Yano-Killing tensors. Since most of the consideration here is independent of the dimension of a manifold, we will not restrict ourselves to spacetime of dimension four. We assume that we are dealing with $n$-dimensional manifold and that this manifold has a metric $g$ (signature of $g$ plays no role).

## 3.1 Basic formulae

Let $\Gamma^\alpha{}_{\mu\nu}$ denotes Christoffel symbols of Levi-Civita connection associated with the metric $g$. We have:

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu}+g_{\beta\nu,\mu}-g_{\mu\nu,\beta}). \tag{3.1}$$

Let $\tilde{g}$ be conformally rescaled metric, i.e. $\tilde{g}_{\mu\nu} := \Omega^2 g_{\mu\nu}$ (and what follows, $\tilde{g}^{\mu\nu} := \Omega^{-2}g^{\mu\nu}$), where $\Omega$ is a certain positive function ($\Omega>0$). We will denote Christoffel symbols of this metric by $\tilde{\Gamma}^\alpha{}_{\mu\nu}$, and covariant derivative associated with them by $\tilde{\nabla}_\mu$. Obviously, for $\tilde{g}_{\mu\nu}$ and $\tilde{\Gamma}^\alpha{}_{\mu\nu}$ we have formula analogous to (3.1). We have:

$$\begin{aligned}\tilde{\Gamma}^\alpha{}_{\mu\nu} &= \frac{1}{2}\tilde{g}^{\alpha\beta}(\tilde{g}_{\beta\mu,\nu}+\tilde{g}_{\beta\nu,\mu}-\tilde{g}_{\mu\nu,\beta})\\ &= \frac{1}{2}\Omega^{-2}g^{\alpha\beta}\left((\Omega^2 g_{\beta\mu})_{,\nu})+(\Omega^2 g_{\beta\nu})_{,\mu}+(\Omega^2 g_{\mu\nu})_{,\beta}\right)\\ &= \Gamma^\alpha{}_{\mu\nu}+\frac{1}{2}\Omega^{-2}g^{\alpha\beta}\left((\Omega^2)_{,\nu}g_{\beta\mu}+(\Omega^2)_{,\mu}g_{\beta\nu}+(\Omega^2)_{,\beta}g_{\mu\nu}\right)\end{aligned}$$

$$= \Gamma^{\alpha}{}_{\mu\nu} + g^{\alpha\beta} \left( U_{,\nu} g_{\beta\mu} + U_{,\mu} g_{\beta\nu} + U_{,\beta} g_{\mu\nu} \right)$$

$$= \Gamma^{\alpha}{}_{\mu\nu} + \delta^{\alpha}{}_{\mu} U_{,\nu} + \delta^{\alpha}{}_{\nu} U_{,\mu} - g^{\alpha\beta} U_{,\beta} g_{\mu\nu} \, , \tag{3.2}$$

where $U := \log \Omega$. Using the formula (3.2) and formulas:

$$\nabla_{\mu} X_{\nu\rho} = X_{\nu\rho,\mu} - X_{\alpha\rho} \Gamma^{\alpha}{}_{\nu\mu} - X_{\nu\alpha} \Gamma^{\alpha}{}_{\rho\mu} \tag{3.3}$$

and

$$\tilde{\nabla}_{\mu} X_{\nu\rho} = X_{\nu\rho,\mu} - X_{\alpha\rho} \tilde{\Gamma}^{\alpha}{}_{\nu\mu} - X_{\nu\alpha} \tilde{\Gamma}^{\alpha}{}_{\rho\mu} \tag{3.4}$$

(which are true for any tensor $X_{\mu\nu}$), we get:

$$\begin{aligned} \tilde{\nabla}_{\mu} X_{\nu\rho} = {} & \nabla_{\mu} X_{\nu\rho} - X_{\mu\rho} U_{,\nu} - X_{\nu\mu} U_{,\rho} - 2 X_{\nu\rho} U_{,\mu} + \\ & + g^{\alpha\beta} U_{,\beta} \left( X_{\alpha\rho} g_{\mu\nu} + X_{\nu\alpha} g_{\mu\rho} \right) . \end{aligned} \tag{3.5}$$

## 3.2 Reducing CYK tensors to Yano tensors

The following theorem has been proved in [13]:

**Theorem 1** *If $Q_{\mu\nu}$ is a CYK tensor for the metric $g_{\mu\nu}$, then $\Omega^3 Q_{\mu\nu}$ is a CYK tensor for the conformally rescaled metric $\Omega^2 g_{\mu\nu}$.*

The situation which we were dealing here and in [13] was the particular one: each considered CYK tensor either was Yano tensor or had the dual one being Yano tensor. We can ask whether this situation is truly particular. Obviously, if we conformally rescale the metric and its CYK tensors (according to the Theorem 2 in [13]), then we get CYK tensors which in general are not Yano tensors, even if they were Yano before the rescaling. However, we can ask if a well chosen conformal factor can bring the situation to one we were dealing with in [13]. In the case of manifold of dimension different than four we cannot define dual CYK tensor. Nevertheless we can ask *whether every CYK tensor can be reduced to Yano tensor by a properly chosen conformal transformation*.

Let $g_{\mu\nu}$ be a metric of a manifold $M$ and $Q_{\mu\nu}$ its CYK tensor. If $\Omega$ is a positive function, then according to the Theorem 1, $\tilde{Q}_{\mu\nu} := \Omega^3 Q_{\mu\nu}$ is a CYK tensor of the metric $\tilde{g}_{\mu\nu} := \Omega^2 g_{\mu\nu}$. The necessary and sufficient condition for $\tilde{Q}_{\mu\nu}$ being Yano tensor is vanishing of $\tilde{\xi}_{\mu}$ defined as follows

$$\tilde{\xi}_{\rho} := \tilde{g}^{\mu\nu} \tilde{\nabla}_{\mu} \tilde{Q}_{\nu\rho} \, . \tag{3.6}$$

¿From the formula (cf. Appendix A of [13])

$$\tilde{\xi}_{\rho} \;\; = \;\; \Omega \left( \xi_{\rho} + (n-1) g^{\mu\nu} Q_{\mu\rho} U_{,\nu} \right) , \tag{3.7}$$

where $\xi_{\rho} := g^{\mu\nu} \nabla_{\mu} Q_{\nu\rho}$, we get that this condition is equivalent to:

$$\xi_{\rho} = (1-n) g^{\mu\nu} Q_{\mu\rho} U_{,\nu} \, , \tag{3.8}$$

where $\xi_\rho := g^{\mu\nu} Q_{\mu\rho;\nu}$ and $U := \log \Omega$. We see that CYK tensor $Q_{\mu\nu}$ can be reduced to Yano tensor if and only if there exist an exact form $\zeta_\mu$ such that

$$\xi^\nu = Q^{\mu\nu} \zeta_\mu \tag{3.9}$$

(namely $\zeta_\mu = (1-n)U_{,\mu}$). It seems that there are no reasons for claiming that for every CYK tensor there exist $\zeta_\mu$ fulfilling the equation (3.9) and moreover that this $\zeta_\mu$ is exact (or even closed). In particular, it is easy to show an example of a CYK tensor for which there are no globally defined form $\zeta_\mu$. Let us consider Minkowski spacetime ($g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ in Cartesian coordinates $x^\mu$). Let us denote by $\mathcal{D}$ a *dilation vector field*:

$$\mathcal{D} := x^\mu \frac{\partial}{\partial x^\mu}\,, \tag{3.10}$$

and by $\mathcal{T}_\mu$, $\mathcal{L}_{\mu\nu}$ generators of Poincare group:

$$\mathcal{T}_\mu := \frac{\partial}{\partial x^\mu}, \quad \mathcal{L}_{\mu\nu} := x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}\,. \tag{3.11}$$

The CYK tensor $Q = \mathcal{D} \wedge \mathcal{T}_0$ has its corresponding vector $\xi$ equal to $\frac{3}{2}\partial_0$. Let us notice that $\mathcal{D}$ vanishes in the point $x^\mu = 0$, $\mu = 0, \ldots, 3$. It means that in this point $Q = 0$ and from the equation (3.9) we get that for any $\zeta$ the field $\xi$ is equal to zero which contradict the fact that $\xi = \frac{3}{2}\partial_0$. Therefore the equation (3.9) cannot be fulfilled everywhere.

We can now check how the situation changes for the case of the Kerr metric:

$$g_{\text{Kerr}} = g_{tt}\,\mathrm{d}t^2 + 2g_{t\phi}\,\mathrm{d}t\,\mathrm{d}\phi + g_{rr}\,\mathrm{d}r^2 + g_{\theta\theta}\,\mathrm{d}\theta^2 + g_{\phi\phi}\,\mathrm{d}\phi^2\,, \tag{3.12}$$

where

$$g_{tt} = -1 + \frac{2mr}{\rho^2}, \quad g_{t\phi} = -\frac{2mra\sin^2\theta}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\triangle}, \quad g_{\theta\theta} = \rho^2,$$

$$g_{\phi\phi} = \sin^2\theta \left( r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\rho^2} \right), \tag{3.13}$$

with

$$\rho^2 = r^2 + a^2\cos^2\theta \quad \text{and} \quad \triangle = r^2 - 2mr + a^2\,, \tag{3.14}$$

and for the Euclidean Taub-NUT metric given by (2.4). In spacetimes (3.12-3.13) and (2.4) we have the following CYK tensors:

$$Q_{\text{Kerr}} = r\sin\theta\,\mathrm{d}\theta \wedge \left[ \left(r^2+a^2\right)\,\mathrm{d}\phi - a\,\mathrm{d}t \right] + a\cos\theta\,\mathrm{d}r \wedge \left( \mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi \right). \tag{3.15}$$

$$*\,Q_{\text{Kerr}} = a\cos\theta\sin\theta\,\mathrm{d}\theta \wedge \left[ \left(r^2+a^2\right)\,\mathrm{d}\phi - a\,\mathrm{d}t \right] + r\,\mathrm{d}r \wedge \left( a\sin^2\theta\,\mathrm{d}\phi - \,\mathrm{d}t \right) \tag{3.16}$$

in Kerr spacetime (see [13]) and

$$Y_{\text{NUT}} = 2m^2\left(\,\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right) \wedge \,\mathrm{d}r + r\,(r+m)\,(r+2m)\sin\theta\,\mathrm{d}\theta \wedge \,\mathrm{d}\phi, \tag{3.17}$$

$$* Y_{\mathrm{NUT}} = 2m\,(m+r)\,(\,\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi) \wedge \,\mathrm{d}r + mr\,(r+2m)\sin\theta\,\mathrm{d}\theta \wedge \,\mathrm{d}\phi. \quad (3.18)$$

for Taub-NUT respectively (cf. 2.6 and 2.8). It turns out that $*Y_{\mathrm{NUT}}$ defined by the formula (2.8) can be reduced to Yano tensor by a properly chosen conformal transformation. For $*Q_{\mathrm{Kerr}}$ defined by the formula (3.16) we can only find $U$ satisfying the equation (3.8) which is not defined on the plane $\theta = \pi/2$. We will discuss this case first.

We denoted by $g_{\mathrm{Kerr}}$ the metric tensor defined by the formulae (3.12)–(3.14). $*Q_{\mathrm{Kerr}}$ given by (3.16) is a CYK tensor of the metric $g_{\mathrm{Kerr}}$, but it is not its Yano tensor, since we have $\chi^\nu := *Q^{\mu\nu}{}_{;\mu} = 3\delta^\nu{}_t$. Rewriting the equation (3.8) for $*Q$ and $\chi$ we get:

$$\chi^\nu = -3 *Q^{\mu\nu} U_{,\mu} \tag{3.19}$$

(obviously indices in $*Q$ were raised with respect to the metric $g_{\mathrm{Kerr}}$). The only non-vanishing components of the tensor $*Q^{\mu\nu}$ are the following:

$$*Q^{\theta t} = \frac{a^2 \sin\theta\cos\theta}{\rho^2}, \quad *Q^{rt} = \frac{r(r^2+a^2)}{\rho^2},$$

$$*Q^{\theta\phi} = \frac{a\cos\theta}{\sin\theta\rho^2} \quad \text{i} \quad *\,Q^{r\phi} = \frac{ar}{\rho^2}.$$

Let us write the components of equation (3.19):

$$1 = -*\,Q^{\mu t} U_{,\mu} = Q^{t\theta} U_{,\theta} + Q^{tr} U_{,r}, \tag{3.20}$$

$$0 = -*\,Q^{\mu r} U_{,\mu} = Q^{rt} U_{,t} + Q^{r\phi} U_{,\phi}, \tag{3.21}$$

$$0 = -*\,Q^{\mu\theta} U_{,\mu} = Q^{\theta t} U_{,t} + Q^{\theta\phi} U_{,\phi}, \tag{3.22}$$

$$0 = -*\,Q^{\mu\phi} U_{,\mu} = Q^{\phi\theta} U_{,\theta} + Q^{\phi r} U_{,r}. \tag{3.23}$$

¿From equations (3.21) and (3.22) we get that

$$-\frac{Q^{r\phi}}{Q^{rt}} U_{,\phi} = U_{,t} = -\frac{Q^{\theta\phi}}{Q^{\phi t}} U_{,\phi}\,.$$

If $U_{,\phi} \neq 0$, then we obtain the following contradiction:

$$\frac{a}{r^2+a^2} = \frac{Q^{r\phi}}{Q^{rt}} = \frac{Q^{\theta\phi}}{Q^{\phi t}} = \frac{1}{a\sin\theta}$$

which implies that $U_{,\phi} = U_{,t} = 0$. Moreover, from equations (3.20) and (3.23) we get

$$1 = (Q^{tr} - Q^{t\theta}\frac{Q^{\phi r}}{Q^{\phi\theta}}) U_{,r} = -rU_{,r}.$$

Therefore $U = -\log r + f(\theta)$, where $f$ is a certain function of one variable. ¿From the equation (3.23) we get the following ODE:

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} = U_{,\theta} = -\frac{Q^{\phi r}}{rQ^{\phi\theta}} = \frac{\sin\theta}{\cos\theta}$$

which maybe easily integrated in closed form: $f = -\log|\cos\theta| + c_1$, where $c_1$ is a certain constant. Finally we get that $U = -\log|r\cos\theta| + c_1$ hence

$$\Omega = \frac{c_2}{r|\cos\theta|},$$

where $c_2 = e^{c_1}$ is a positive constant. Let us observe that $U$ and $\Omega$ are not defined (or singular) on the plane $\theta = \frac{\pi}{2}$. Direct computation shows that tensor $\Omega^3 * Q$ is indeed Yano tensor of the metric $\Omega^2 g_{\text{Kerr}}$. Tensor $\Omega^3 Q$ is no longer Yano tensor of the conformally rescaled metric (although obviously it is its CYK tensor), since we have $\tilde{\xi}_{\text{Kerr}} = \frac{3a}{c_2}\partial_t + \frac{3}{c_2}\partial_\phi$ (where $\tilde{\xi}_{\text{Kerr}}$ is defined by (3.6)). The vector field $\tilde{\xi}_{\text{Kerr}}$ is Killing vector field of the metric $\Omega^2 g_{\text{Kerr}}$. It seems that it is only a coincidence, since $\Omega^2 g_{\text{Kerr}}$ is no longer an Einstein metric.

Finally, we discuss the case of CYK tensor of Taub-NUT metric. Let $g_{\text{NUT}}$ denote this metric defined by the formula (2.4). Tensor $*Y_{\text{NUT}}$ defined by the formula (2.8) is its CYK tensor but is not its Yano tensor, since $\chi^\nu := *Y^{\mu\nu}{}_{;\mu} = -\frac{3}{2m}\delta^\nu{}_\psi$. In order to find conformal factor $\Omega'$, which reduces $*Y$ to Yano tensor, let us rewrite the equation (3.8) for $*Y$, $\chi$ and $U' := \log\Omega'$. We have:

$$\chi^\nu = -3 * Y^{\mu\nu} U'_{,\mu} \tag{3.24}$$

The only non-vanishing components of the tensor $*Y^{\mu\nu}$ are the following:

$$*Y^{\psi\theta} = \frac{m\cos\theta}{r(r+2m)\sin\theta}, \quad *Y^{\psi r} = \frac{m+4}{2m}, \quad *Y^{\theta\phi} = \frac{m}{r(r+2m)\sin\theta}.$$

Let us write out the components of the equation (3.24):

$$1 = 2m * Y^{\mu\psi} U'_{,\mu} = 2m * Y^{\theta\psi} U'_{,\theta} + 2m * Y^{r\psi} U'_{,r}\,, \tag{3.25}$$
$$0 = 2m * Y^{\mu r} U'_{,\mu} = 2m * Y^{\psi r} U'_{,\psi}\,, \tag{3.26}$$
$$0 = 2m * Y^{\mu\theta} U'_{,\mu} = 2m * Y^{\psi\theta} U'_{,\psi} + 2m * Y^{\phi\theta} U'_{,\phi}\,, \tag{3.27}$$
$$0 = 2m * Y^{\mu\phi} U'_{,\mu} = 2m * Y^{\theta\phi} U'_{,\theta}\,, \tag{3.28}$$

Equations (3.26)–(3.28) imply that $U'_{,\psi} = U'_{,\theta} = U'_{,\phi} = 0$, that is $U' = U'(r)$. Using this result and the equation (3.25) we get:

$$1 = \frac{\mathrm{d}U'}{\mathrm{d}r} 2m * Y^{r\psi} = -(m+r)\frac{\mathrm{d}U'}{\mathrm{d}r},$$

that is $U' = -\log(m+r) + c_3$, where $c_3$ is a certain constant. Finally:

$$\Omega' = \frac{c_4}{m+r},$$

where $c_4 = e^{c_3}$ is a positive constant. Again direct computation shows that $\Omega'^3 * Y$ is a Yano tensor of the metric $\Omega'^2 g_{\text{NUT}}$. The tensor $\Omega'^3 Y$ (where $Y$ is defined by the formula (2.6)) is a CYK tensor of the conformally rescaled metric, for which we have: $\tilde{\xi}_{\text{NUT}} = \frac{3}{2c_4}\partial_\psi$. Although $\Omega'^2 g_{\text{NUT}}$ is no longer an Einstein metric, a vector $\tilde{\xi}_{\text{NUT}}$ is its Killing vector.

Considerations in this Section show that there is no unique answer to the question if we can reduce CYK to Yano via conformal rescaling, for Kerr the answer is negative but for Euclidean Taub-NUT is positive.

# References

[1] G. Bergqvist, P. Lankinen, *Unique characterization of the Bel–Robinson tensor*, Classical and Quantum Gravity 21, (2004) 3489–3503; G. Bergqvist, I. Eriksson and J.M.M. Senovilla, *New electromagnetic conservation laws*, Classical and Quantum Gravity 20, (2003) 2663–68

[2] M.A.G. Bonilla and J.M.M. Senovilla, *Some Properties of the Bel and Bel–Robinson Tensors* General Relativity and Gravitation 29, (1997) 91–116

[3] D. Christodoulou and S. Klainerman, *Asymptotic Properties of Linear Field Equations in Minkowski Space*, Communications on Pure and Applied Mathematics 43, (1990) 137–199

[4] S.R. Douglas, *Letter: Review of the Definitions of the Bel and Bel-Robinson Tensors*, General Relativity and Gravitation 35, (2003) 1691–97

[5] G.W. Gibbons, R.H. Rietdijk and J.W. van Holten, *SUSY in the sky*, Nucl. Phys. B 404, (1993) 42–64, http://arxiv.org/abs/hep-th/9303112

[6] E.N. Glass and M.G. Naber, *Gravitational mass anomaly*, Journal of Mathematical Physics 35, (1994) 4178–83

[7] J.N. Goldberg, *Conserved quantities at spatial and null infinity: The Penrose potential*, Phys. Rev. D 41, (1990) 410–417

[8] J.W. van Holten, *Phys. Lett.* B **342**, (1995) 47-52

[9] J. Jezierski, *The Relation between Metric and Spin–2 Formulations of Linearized Einstein Theory*, Gen. Rel. and Grav. 27, (1995) 821–843, http://arxiv.org/abs/gr-qc/9411066

[10] J. Jezierski, *Conformal Yano–Killing tensors and asymptotic CYK tensors for the Schwarzschild metric*, Classical and Quantum Gravity 14, (1997) 1679–1688, http://arxiv.org/abs/hep-th/9411074

[11] J. Jezierski, *Gauge-invariant formulation of the linearized Einstein equations on the Schwarzschild background*, in Current Topics in Math. Cosmology, eds M. Rainer and H-J Schmidt, World Scientific (1998); *Energy and angular momentum of the weak gravitational waves on the Schwarzschild background – quasilocal gauge-invariant formulation*, General Relativity and Gravitation 31, (1999) 1855–1890, http://arxiv.org/abs/gr-qc/9801068

[12] J. Jezierski, *CYK Tensors, Maxwell Field and Conserved Quantities for Spin-2 Field*, Classical and Quantum Gravity 19, (2002) 4405–4429, http://arxiv.org/abs/gr-qc/0211039

[13] J. Jezierski, M. Łukasik, *Conformal Yano-Killing tensor for the Kerr metric and conserved quantities*, Class. Quantum Grav. 23, (2006) 2895–2918, http://arxiv.org/abs/gr-qc/0510058

[14] A. Moroianu, U. Semmelmann, *Twistor forms on Kähler manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) II (2003), 823–845, `http://arxiv.org/abs/math.DG/0204322`

[15] R. Penrose, *Quasi-local Mass and Angular Momentum in General Relativity*, Proc. Roy. Soc. Lond. A381, (1982) 53–62

[16] R. Penrose and W. Rindler, *Spinors and Space-time*, Cambridge University Press, Vol. 2, p. 396 (Cambridge 1986)

[17] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Mathematische Zeitschrift 245, (2003) 503–527

[18] S.E. Stepanov, *The Vector Space of Conformal Killing Forms on a Riemannian Manifold*, Journal of Mathematical Sciences 110, (2002) 2892–2906; *On conformal Killing 2-form of the electromagnetic field*, Journal of Geometry and Physics 33, (2000) 191–209

[19] H. Stephani et al., *Exact solutions of Einstein's field equations*, 2nd ed., University Press (Cambridge 2003)

[20] S. Tachibana, *On conformal Killing tensor in a Riemannian space*, Tohoku Math. J. (2) 21, (1969) 56–64; T. Kashiwada, *On conformal Killing tensor*, Natur. Sci. Rep. Ochanomizu Univ. 19, (1968) 67–74; S. Tachibana and T. Kashiwada, *On the integrability of Killing–Yano's equation*, J. Math. Soc. Japan 21, (1969) 259–265

[21] K. Yano, *Some remarks on tensor fields and curvature*, Ann. Math. 55, (1952) 328–347